

# DEFORMATIONS OF THE GENERALISED PICARD BUNDLE

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**ABSTRACT.** Let  $X$  be a nonsingular algebraic curve of genus  $g \geq 3$ , and let  $\mathcal{M}_\xi$  denote the moduli space of stable vector bundles of rank  $n \geq 2$  and degree  $d$  with fixed determinant  $\xi$  over  $X$  such that  $n$  and  $d$  are coprime. We assume that if  $g = 3$  then  $n \geq 4$  and if  $g = 4$  then  $n \geq 3$ , and suppose further that  $n_0, d_0$  are integers such that  $n_0 \geq 1$  and  $nd_0 + n_0d > nn_0(2g - 2)$ . Let  $E$  be a semistable vector bundle over  $X$  of rank  $n_0$  and degree  $d_0$ . The generalised Picard bundle  $\mathcal{W}_\xi(E)$  is by definition the vector bundle over  $\mathcal{M}_\xi$  defined by the direct image  $p_{\mathcal{M}_\xi*}(\mathcal{U}_\xi \otimes p_X^*E)$  where  $\mathcal{U}_\xi$  is a universal vector bundle over  $X \times \mathcal{M}_\xi$ . We obtain an inversion formula allowing us to recover  $E$  from  $\mathcal{W}_\xi(E)$  and show that the space of infinitesimal deformations of  $\mathcal{W}_\xi(E)$  is isomorphic to  $H^1(X, \text{End}(E))$ . This construction gives a locally complete family of vector bundles over  $\mathcal{M}_\xi$  parametrised by the moduli space  $\mathcal{M}(n_0, d_0)$  of stable bundles of rank  $n_0$  and degree  $d_0$  over  $X$ . If  $(n_0, d_0) = 1$  and  $\mathcal{W}_\xi(E)$  is stable for all  $E \in \mathcal{M}(n_0, d_0)$ , the construction determines an isomorphism from  $\mathcal{M}(n_0, d_0)$  to a connected component  $\mathcal{M}^0$  of a moduli space of stable sheaves over  $\mathcal{M}_\xi$ . This applies in particular when  $n_0 = 1$ , in which case  $\mathcal{M}^0$  is isomorphic to the Jacobian  $J$  of  $X$  as a polarised variety. The paper as a whole is a generalisation of results of Kempf and Mukai on Picard bundles over  $J$ .

## 1. INTRODUCTION

Let  $X$  be a connected nonsingular projective algebraic curve of genus  $g \geq 2$  defined over the complex numbers. Let  $J$  denote the Jacobian (Picard variety) of  $X$  and  $J^d$  the variety of line bundles of degree  $d$  over  $X$ ; thus in particular  $J^0 = J$ . Suppose  $d \geq 2g - 1$  and let  $\mathcal{L}$  be a Poincaré (universal) bundle over  $X \times J^d$ . If we denote by  $p_J$  the natural projection from  $X \times J^d$  to  $J^d$ , the direct image  $p_{J*}\mathcal{L}$  is then locally free and is called the *Picard bundle* of degree  $d$ .

These bundles have been investigated by a number of authors over at least the last 40 years. It may be noted that the projective bundle corresponding to  $p_{J*}\mathcal{L}$  can be identified with the  $d$ -fold symmetric product  $S^d(X)$ . Picard bundles were studied in this light by A. Mattuck [12, 13] and I. G. Macdonald [14] among others; both Mattuck and Macdonald gave formulae for their Chern classes. Somewhat later R. C. Gunning [7, 8] gave a more analytic treatment involving theta-functions. Later still, and of especial relevance to us, G. Kempf [10] and S. Mukai [15] independently studied the deformations of the Picard bundle; the problem then is to obtain an inversion

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formula showing that all deformations of  $p_{J*}\mathcal{L}$  arise in a natural way. Kempf and Mukai proved that  $p_{J*}\mathcal{L}$  is simple and that, if  $X$  is not hyperelliptic, the space of infinitesimal deformations of  $p_{J*}\mathcal{L}$  has dimension given by

$$\dim H^1(J^d, \text{End}(p_{J*}\mathcal{L})) = 2g.$$

Moreover, all the infinitesimal deformations arise from genuine deformations. In fact there is a complete family of deformations of  $p_{J*}\mathcal{L}$  parametrised by  $J \times \text{Pic}^0(J^d)$ , the two factors corresponding respectively to translations in  $J^d$  and deformations of  $\mathcal{L}$  ([10, §9], [15, Theorem 4.8]). (The deformations of  $\mathcal{L}$  are given by  $\mathcal{L} \mapsto \mathcal{L} \otimes p_J^*L$  for  $L \in \text{Pic}^0(J^d)$ .) Since  $J$  is a principally polarised abelian variety and  $J^d \cong J$ ,  $\text{Pic}^0(J^d)$  can be identified with  $J$  (strictly speaking  $\text{Pic}^0(J^d)$  is the dual abelian variety, but the principal polarisation allows the identification).

Mukai's paper [15] was set in a more general context involving a transform which provides an equivalence between the derived category of the category of  $\mathcal{O}_A$ -modules over an abelian variety  $A$  and the corresponding derived category on the dual abelian variety  $\hat{A}$ . This technique has come to be known as the Fourier–Mukai transform and has proved very useful in studying moduli spaces of sheaves on abelian varieties and on some other varieties.

Our object in this paper is to generalise the results of Kempf and Mukai on deformations of Picard bundles to the moduli spaces of higher rank vector bundles over  $X$  with fixed determinant. In particular we obtain an inversion formula for our generalised Picard bundles and compute their spaces of infinitesimal deformations. We also identify a family of deformations which is locally complete and frequently globally complete as well. The construction of the generalised Picard bundles together with the inversion formula can be seen as a type of Fourier–Mukai transform.

We fix a holomorphic line bundle  $\xi$  over  $X$  of degree  $d$ . Let  $\mathcal{M}_\xi := \mathcal{M}_\xi(n, d)$  be the moduli space of stable vector bundles  $F$  over  $X$  with  $\text{rank}(F) = n \geq 2$ ,  $\deg(F) = d$  and  $\bigwedge^n F = \xi$ . We assume that  $n$  and  $d$  are coprime, ensuring the smoothness and completeness of  $\mathcal{M}_\xi$ , and that  $g \geq 3$ . We assume also that if  $g = 3$  then  $n \geq 4$  and if  $g = 4$  then  $n \geq 3$ . The case  $g = 2$  together with the three special cases  $g = 3$  with  $n = 2, 3$  and  $g = 4$  with  $n = 2$  are omitted in our main results since the method of proof does not cover these cases.

It is known that there is a universal vector bundle over  $X \times \mathcal{M}_\xi$ . Two such universal bundles differ by tensoring with the pullback of a line bundle on  $\mathcal{M}_\xi$ . However, since  $\text{Pic}(\mathcal{M}_\xi) = \mathbb{Z}$ , it is possible to choose canonically a universal bundle. Let  $l$  be the smallest positive number such that  $ld \equiv 1 \pmod n$ . There is a unique universal vector bundle  $\mathcal{U}_\xi$  over  $X \times \mathcal{M}_\xi$  such that  $\bigwedge^n \mathcal{U}_\xi|_{\{x\} \times \mathcal{M}_\xi} = \Theta^{\otimes l}$  [18], where  $x \in X$  and  $\Theta$  is the ample generator of  $\text{Pic}(\mathcal{M}_\xi)$ . Henceforth, by a universal bundle we will always mean this canonical one. We denote by  $p_X$  and  $p_{\mathcal{M}_\xi}$  the natural projections of  $X \times \mathcal{M}_\xi$  onto the two factors.

Now suppose that  $n_0$  and  $d_0$  are integers with  $n_0 \geq 1$  and

$$(1) \quad nd_0 + n_0d > nn_0(2g - 2).$$

For any semistable vector bundle  $E$  of rank  $n_0$  and degree  $d_0$  over  $X$ , let

$$\mathcal{W}_\xi(E) := p_{\mathcal{M}_\xi*}(\mathcal{U}_\xi \otimes p_X^* E)$$

be the direct image. The assumption (1) ensures that  $\mathcal{W}_\xi(E)$  is a locally free sheaf on  $\mathcal{M}_\xi$  and all the higher direct images of  $\mathcal{U}_\xi \otimes p_X^* E$  vanish. The rank of  $\mathcal{W}_\xi(E)$  is  $nd_0 + n_0d + nn_0(1-g)$  and  $H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(E)) \cong H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E)$ . We shall refer to the bundles  $\mathcal{W}_\xi(E)$  as *generalised Picard bundles*.

Our first main result is an inversion formula for this construction.

**Theorem 5.1.** *Suppose that (1) holds and that  $E$  is a semistable bundle of rank  $n_0$  and degree  $d_0$ . Then*

$$E \cong \mathcal{R}^1 p_{X*}(p_{\mathcal{M}_\xi}^* \mathcal{W}_\xi(E) \otimes \mathcal{U}_\xi^* \otimes p_X^* K_X).$$

Following this, we show that, if  $E$  is simple as well as semistable, then  $\mathcal{W}_\xi(E)$  is simple (Corollary 6.2). Moreover

**Theorem 6.3.** *Suppose that (1) holds. For any semistable bundle  $E$  of rank  $n_0$  and degree  $d_0$ , the space of infinitesimal deformations of the vector bundle  $\mathcal{W}_\xi(E)$ , namely  $H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(E)))$ , is canonically isomorphic to  $H^1(X, \text{End} E)$ . In particular, if  $E$  is also simple,*

$$\dim H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(E))) = n_0^2(g-1) + 1.$$

In the special case where  $n = 2$  and  $E = \mathcal{O}_X$ , this was proved by V. Balaji and P. A. Vishwanath in [1] using a construction of M. Thaddeus [19]. For all  $n$ , it is known that  $\mathcal{W}_\xi(\mathcal{O}_X)$  is simple [5] and indeed that it is stable (with respect to the unique polarisation of  $\mathcal{M}_\xi$ ) [4]; in fact the proof of stability generalises easily to show that  $\mathcal{W}_\xi(L)$  is stable for any line bundle  $L$  for which (1) holds. In this context, note that Y. Li [11] has proved a stability result for Picard bundles over the non-fixed determinant moduli space  $\mathcal{M}(n, d)$ , but this does not imply the result for  $\mathcal{M}_\xi$ .

If  $(n_0, d_0) = 1$ , we can consider the bundles  $\{\mathcal{W}_\xi(E)\}$  as a family of bundles over  $\mathcal{M}_\xi$ , parametrized by  $\mathcal{M}(n_0, d_0)$ . We prove that this family is locally complete, i. e. that the infinitesimal deformation map

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(E)))$$

is an isomorphism for all  $E$  (Theorem 7.1). If all the bundles  $\mathcal{W}_\xi(E)$  are stable, the family is also globally complete (Theorem 7.3). Finally, in the case  $n_0 = 1$ , we obtain an isomorphism of polarised varieties between  $J$  and a connected component of a moduli space of bundles over  $\mathcal{M}_\xi$  (Theorem 7.4), which in turn leads to a Torelli theorem (Corollary 7.5).

The layout of the paper is as follows. In sections 2, 3 and 4, we obtain cohomological results. The techniques are quite similar to those of Kempf and Mukai except for our use of Hecke transformations; however our calculations are more complicated since we

cannot exploit the special properties of abelian varieties. In sections 5, 6 and 7, we then use these results to obtain our main theorems.

*Notation and assumptions.* We work throughout over the complex numbers and suppose that  $X$  is a connected nonsingular projective algebraic curve of genus  $g \geq 3$ . We suppose that  $n \geq 2$  and that if  $g = 3$  then  $n \geq 4$  and if  $g = 4$  then  $n \geq 3$ . We assume moreover that  $(n, d) = 1$  and that (1) holds. In general, we denote the natural projections of  $X \times Y$  onto its factors by  $p_X, p_Y$ . For a variety  $X \times Y \times Z$ , we denote by  $p_i$  ( $i = 1, 2, 3$ ) the projection onto the  $i$ -th factor and by  $p_{ij}$  the projection onto the Cartesian product of the  $i$ -th and the  $j$ -th factors. Finally, for any  $x \in X$ , we denote by  $\mathcal{U}_x$  the bundle over  $\mathcal{M}_\xi$  obtained by restricting  $\mathcal{U}_\xi$  to  $\{x\} \times \mathcal{M}_\xi$ .

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## 2. COHOMOLOGY OF $\mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*$

Our principal object in this section and the two following sections is to compute the cohomology groups  $H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*)$  for  $i = 0, 1$ , where  $E_1$  and  $E_2$  are semistable bundles of rank  $n_0$  and degree  $d_0$  satisfying (1).

**Proposition 2.1.** *Suppose that (1) holds and  $E_1$  and  $E_2$  are semistable bundles of rank  $n_0$  and degree  $d_0$ . Then*

$$\begin{aligned} H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*) &\cong H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{W}_\xi(E_2)^*) \\ &\cong H^{i+1}(X \times \mathcal{M}_\xi \times X, p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^* \otimes p_3^* E_2^* \otimes p_3^* K_X) \end{aligned}$$

for  $i \geq 0$ , where  $K_X$  is the canonical line bundle over  $X$ .

*Proof.* Recall first that, if  $E$  and  $F$  are semistable, then so is  $F \otimes E$ . It then follows from (1) that  $H^1(X, \mathcal{U}_\xi|_{X \times \{v\}} \otimes E_1) = 0$  for all  $v \in \mathcal{M}_\xi$ . Using the projection formula and the Leray spectral sequence we have

$$H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{W}_\xi(E_2)^*) \cong H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*).$$

This proves the first isomorphism.

In the same way, (1) gives

$$H^0(X, (\mathcal{U}_\xi|_{X \times \{v\}})^* \otimes E_2^* \otimes K_X) \cong H^1(X, \mathcal{U}_\xi|_{X \times \{v\}} \otimes E_2)^* = 0.$$

Consequently, the projection formula gives

$$\mathcal{R}^i p_{12*}(p_{23}^*(\mathcal{U}_\xi^*) \otimes p_3^* E_2^* \otimes p_3^* K_X) = 0$$

for  $i \neq 1$ , and we have by relative Serre duality

$$\mathcal{R}^1 p_{12*}(p_{23}^*(\mathcal{U}_\xi^*) \otimes p_3^* E_2^* \otimes p_3^* K_X) \cong p_{\mathcal{M}_\xi}^* \mathcal{W}_\xi(E_2)^*.$$

Finally, using the projection formula and the Leray spectral sequence, it follows that

$$H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{W}_\xi(E_2)^*)$$

$$\cong H^{i+1}(X \times \mathcal{M}_\xi \times X, p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^* \otimes p_3^* E_2^* \otimes p_3^* K_X).$$

for  $i \geq 0$ . This completes the proof.  $\square$

**Remark 2.2.** Proposition 2.1 can be formulated in a more general context. Let  $V_1, V_2$  be flat families of vector bundles over  $X$  parametrised by a complete irreducible variety  $Y$  such that for each  $y \in Y$  we have  $H^1(X, V_i|_{X \times \{y\}}) = 0$  for  $i = 1, 2$ . Under this assumption

$$H^i(Y, p_{Y*} V_1 \otimes (p_{Y*} V_2)^*) \cong H^{i+1}(X \times Y \times X, p_{12}^* V_1 \otimes p_{23}^* V_2^* \otimes p_3^* K_X).$$

The proof is the same as for Proposition 2.1.

We can now state the key result which enables us to calculate the cohomology groups in which we are interested. We denote by  $\mathcal{R}^i$  the  $i$ -th direct image of  $p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*$  for the projection  $p_3$ , that is

$$\mathcal{R}^i := \mathcal{R}^i p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*).$$

**Proposition 2.3.** *For  $i \geq 0$ , there exists an exact sequence*

$$0 \longrightarrow H^1(X, \mathcal{R}^i \otimes E_2^* \otimes K_X) \longrightarrow H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*)$$

$$(2) \quad \longrightarrow H^0(X, \mathcal{R}^{i+1} \otimes E_2^* \otimes K_X) \longrightarrow 0.$$

*Proof.* Since  $\dim X = 1$ , the Leray spectral sequence for  $p_3$  gives

$$0 \longrightarrow H^1(X, \mathcal{R}^i \otimes E_2^* \otimes K_X) \longrightarrow H^{i+1}(X \times \mathcal{M}_\xi \times X, p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^* \otimes p_3^* E_2^* \otimes p_3^* K_X)$$

$$(3) \quad \longrightarrow H^0(X, \mathcal{R}^{i+1} \otimes E_2^* \otimes K_X) \longrightarrow 0.$$

The result now follows at once from Proposition 2.1.  $\square$

In order to use this proposition, we must compute the  $\mathcal{R}^i$  for  $i \leq 2$ . We can already prove

**Proposition 2.4.**  $\mathcal{R}^0 = 0$ .

*Proof.* Note that for any  $x \in X$

$$(4) \quad H^0(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) \cong H^0(X, E_1 \otimes p_{X*}(\mathcal{U}_\xi \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)).$$

From [3] and [16] we know that for generic  $y \in X$ , the two vector bundles  $\mathcal{U}_y$  and  $\mathcal{U}_x$  are non-isomorphic and stable. Hence  $H^0(\mathcal{M}_\xi, \mathcal{U}_y \otimes \mathcal{U}_x^*) = 0$ . This implies that

$$(5) \quad p_{X*}(\mathcal{U}_\xi \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) = 0.$$

So (4) gives

$$\mathcal{R}^0 := p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*) = 0$$

and the proof is complete.  $\square$

In the next two sections we will use Hecke transformations and a diagonal argument to show that  $R^2 = 0$  and to compute  $R^1$ .

## 3. THE HECKE TRANSFORMATION

In this section we will use Hecke transformations to compute the cohomology groups  $H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)$  for any  $x \in X$ . The details of the Hecke transformation and its properties can be found in [16, 17]. We will briefly describe it and note those properties that will be needed here.

Fix a point  $x \in X$ . Let  $\mathbb{P}(\mathcal{U}_x)$  denote the projective bundle over  $\mathcal{M}_\xi$  consisting of lines in  $\mathcal{U}_x$ . If  $f$  denotes the natural projection of  $\mathbb{P}(\mathcal{U}_x)$  to  $\mathcal{M}_\xi$  and  $\mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(-1)$  the tautological line bundle then

$$f_* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) \cong \mathcal{U}_x^*,$$

and  $\mathcal{R}^j f_* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) = 0$  for all  $j > 0$ . From the commutative diagram

$$\begin{array}{ccc} X \times \mathbb{P}(\mathcal{U}_x) & \xrightarrow{\text{Id}_X \times f} & X \times \mathcal{M}_\xi \\ p_{\mathbb{P}(\mathcal{U}_x)} \downarrow & & \downarrow p_{\mathcal{M}} \\ \mathbb{P}(\mathcal{U}_x) & \xrightarrow{f} & \mathcal{M}_\xi \end{array}$$

and the base change theorem, we deduce that

$$\begin{aligned} & H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) \\ (6) \quad & \cong H^i(X \times \mathbb{P}(\mathcal{U}_x), (\text{Id}_X \times f)^*(\mathcal{U}_\xi \otimes p_X^* E_1) \otimes p_{\mathbb{P}(\mathcal{U}_x)}^* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)) \end{aligned}$$

for all  $i$ .

Moreover, since  $p_{\mathbb{P}(\mathcal{U}_x)*}(\text{Id}_X \times f)^*(\mathcal{U}_\xi \otimes p_X^* E_1) \cong f^* \mathcal{W}_\xi(E_1)$ , there is a canonical isomorphism

$$\begin{aligned} & H^i(X \times \mathbb{P}(\mathcal{U}_x), (\text{Id}_X \times f)^*(\mathcal{U}_\xi \otimes p_X^* E_1) \otimes p_{\mathbb{P}(\mathcal{U}_x)}^* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)) \\ (7) \quad & \cong H^i(\mathbb{P}(\mathcal{U}_x), f^* \mathcal{W}_\xi(E_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)) \end{aligned}$$

for all  $i$ .

To compute the cohomology groups  $H^i(\mathbb{P}(\mathcal{U}_x), f^* \mathcal{W}_\xi(E_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1))$  we use Hecke transformations.

A point in  $\mathbb{P}(\mathcal{U}_x)$  represents a stable vector bundle  $F$  and a line  $l$  in the fibre  $F_x$  at  $x$ , or equivalently a non-trivial exact sequence

$$(8) \quad 0 \longrightarrow F \longrightarrow F' \longrightarrow \mathbb{C}_x \longrightarrow 0$$

determined up to a scalar multiple; here  $\mathbb{C}_x$  denotes the torsion sheaf supported at  $x$  with stalk  $\mathbb{C}$ . The sequences (8) fit together to form a universal sequence

$$(9) \quad 0 \longrightarrow (\text{Id}_X \times f)^* \mathcal{U}_\xi \otimes p_{\mathbb{P}(\mathcal{U}_x)}^* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) \longrightarrow \mathcal{F} \longrightarrow p_X^* \mathbb{C}_x \longrightarrow 0$$

on  $X \times \mathbb{P}(\mathcal{U}_x)$ . If  $\eta$  denotes the line bundle  $\xi \otimes \mathcal{O}_X(x)$  over  $X$  and  $\mathcal{M}_\eta$  the moduli space of stable bundles  $\mathcal{M}_\eta(n, d+1)$  then from (8) and (9) we get a rational map

$$\gamma : \mathbb{P}(\mathcal{U}_x) \dashrightarrow \mathcal{M}_\eta$$

which sends any pair  $(F, l)$  to  $F'$ . This map is not everywhere defined since the bundle  $F'$  in (8) need not be stable.

Our next object is to find a Zariski-open subset  $Z$  of  $\mathcal{M}_\eta$ , over which  $\gamma$  is defined and is a projective fibration, such that the complement of  $Z$  in  $\mathcal{M}_\eta$  has codimension at least 4. The construction and calculations are similar to those of [16, Proposition 6.8], but our results do not seem to follow directly from that proposition.

As in [16, §8] or [17, §5], we define a bundle  $F'$  to be  $(0, 1)$ -stable if, for every proper subbundle  $G$  of  $F'$ ,

$$\frac{\deg G}{\operatorname{rk} G} < \frac{\deg F' - 1}{\operatorname{rk} F'}.$$

Clearly every  $(0, 1)$ -stable bundle is stable. We denote by  $Z$  the subset of  $\mathcal{M}_\eta$  consisting of  $(0, 1)$ -stable bundles.

**Lemma 3.1.** (i)  $Z$  is a Zariski-open subset of  $\mathcal{M}_\eta$  whose complement has codimension at least 4.

(ii)  $\gamma$  is a projective fibration over  $Z$  and  $\gamma^{-1}(Z)$  is a Zariski-open subset of  $\mathbb{P}(\mathcal{U}_x)$  whose complement has codimension at least 4.

*Proof.* (i) The fact that  $Z$  is Zariski-open is standard (see [17, Proposition 5.3]).

The bundle  $F' \in \mathcal{M}_\eta$  of rank  $n$  and degree  $d + 1$  fails to be  $(0, 1)$ -stable if and only if it has a subbundle  $G$  of rank  $r$  and degree  $e$  such that  $ne \geq r((d + 1) - 1)$ , i.e.,

$$(10) \quad rd \leq ne.$$

By considering the extensions

$$0 \longrightarrow G \longrightarrow F' \longrightarrow H \longrightarrow 0,$$

we can estimate the codimension of  $\mathcal{M}_\eta - Z$  and show that it is at least

$$(11) \quad \delta = r(n - r)(g - 1) + (ne - r(d + 1))$$

(compare the proof of [17, Proposition 5.4]). Note that, since  $(n, d) = 1$ , (10) implies that  $rd \leq ne - 1$ . Given that  $g \geq 3$ , we see that  $\delta < 4$  only if  $g = 3$ ,  $n = 2, 3$  or  $g = 4$ ,  $n = 2$ . These are exactly the cases that were excluded in the introduction.

(ii)  $\gamma^{-1}(Z)$  consists of all pairs  $(F, l)$  for which the bundle  $F'$  in (8) is  $(0, 1)$ -stable. As in (i), this is a Zariski-open subset. It follows at once from (10) that, if  $F'$  is  $(0, 1)$ -stable, then  $F$  is stable. So, if  $F' \in Z$ , it follows from (8) that  $\gamma^{-1}(F')$  can be identified with the projective space  $\mathbb{P}(F_x'^*)$ . Using the universal projective bundle on  $X \times \mathcal{M}_\eta$ , we see that  $\gamma^{-1}(Z)$  is a projective fibration over  $Z$  (not necessarily locally trivial).

Suppose now that  $(F, l)$  belongs to the complement of  $\gamma^{-1}(Z)$  in  $\mathbb{P}(\mathcal{U}_x)$ . This means that the bundle  $F'$  in (8) is not  $(0, 1)$ -stable and therefore possesses a subbundle  $G$  satisfying (10). If  $G \subset F$ , this contradicts the stability of  $F$ . So there exists an exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow \mathbb{C}_x \longrightarrow 0$$

with  $G'$  a subbundle of  $F$  of rank  $r$  and degree  $e - 1$ . Moreover, since  $G$  is a subbundle of  $F$ ,  $G'_x$  must contain the line  $l$ . For fixed  $r, e$ , these conditions determine a subvariety

of  $\mathbb{P}(\mathcal{U}_x)$  of dimension at most

$$(r^2(g-1)+1) + ((n-r)^2(g-1)+1) - g + (r-1) + ((g-1)r(n-r) + (rd - n(e-1)) - 1).$$

Since  $\dim \mathbb{P}(\mathcal{U}_x) = n^2(g-1) - g + n$ , a simple calculation shows that the codimension is at least the number  $\delta$  given by (11). As in (i), this gives the required result.  $\square$

By Lemma 3.1(ii) and a Hartogs-type theorem (see [9, Theorem 3.8 and Proposition 1.11]) we have an isomorphism

$$(12) \quad H^i(\mathbb{P}(\mathcal{U}_x), f^*\mathcal{W}_\xi(E_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)) \cong H^i(\gamma^{-1}(Z), f^*\mathcal{W}_\xi(E_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)|_{\gamma^{-1}(Z)})$$

for  $i \leq 2$ .

Now let  $F' \in Z$ . As in the proof of Lemma 3.1, we identify  $\gamma^{-1}(F')$  with  $\mathbb{P}(F'_x)$  and denote it by  $\mathbb{P}$ . On  $X \times \mathbb{P}$  there is a universal exact sequence

$$(13) \quad 0 \longrightarrow \mathcal{E} \longrightarrow p_X^*F' \longrightarrow p_{\mathbb{P}}^*\mathcal{O}_{\mathbb{P}}(1) \otimes p_X^*\mathbb{C}_x \longrightarrow 0.$$

The restriction of (13) to any point of  $\mathbb{P}$  is isomorphic to the corresponding sequence (8).

**Proposition 3.2.** *Let  $\mathcal{F}$  be defined by the universal sequence (9). Then*

$$\mathcal{F}|_{X \times \mathbb{P}} \cong p_X^*F' \otimes p_{\mathbb{P}}^*\mathcal{O}_{\mathbb{P}}(-1).$$

*Proof.* Restricting (9) to  $X \times \mathbb{P}$  gives

$$0 \longrightarrow (\text{Id}_X \times f)^*\mathcal{U}_\xi \otimes p_{\mathbb{P}(\mathcal{U}_x)}^*\mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)|_{X \times \mathbb{P}} \longrightarrow \mathcal{F}|_{X \times \mathbb{P}} \longrightarrow p_X^*\mathbb{C}_x \longrightarrow 0.$$

This must coincide with the universal sequence (13) up to tensoring by some line bundle lifted from  $\mathbb{P}$ . The result follows.  $\square$

Next we tensor (9) by  $p_X^*E_1$ , restrict it to  $X \times \gamma^{-1}(Z)$  and take the direct image on  $\gamma^{-1}(Z)$ . This gives

$$(14) \quad 0 \longrightarrow f^*\mathcal{W}_\xi(E_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)|_{\gamma^{-1}(Z)} \longrightarrow p_{\mathbb{P}(\mathcal{U}_x)*}(\mathcal{F} \otimes p_X^*E_1)|_{\gamma^{-1}(Z)} \longrightarrow \mathcal{O}_{\gamma^{-1}(Z)}^{\oplus n_0} \longrightarrow 0.$$

**Proposition 3.3.**  *$R_{\gamma*}^i(p_{\mathbb{P}(\mathcal{U}_x)*}(\mathcal{F} \otimes p_X^*E_1)|_{\gamma^{-1}(Z)}) = 0$  for all  $i$ .*

*Proof.* It is sufficient to show that  $p_{\mathbb{P}(\mathcal{U}_x)*}(\mathcal{F} \otimes p_X^*E_1)|_{\mathbb{P}}$  has trivial cohomology. By Proposition 3.2

$$\begin{aligned} p_{\mathbb{P}(\mathcal{U}_x)*}(\mathcal{F} \otimes p_X^*E_1)|_{\mathbb{P}} &\cong p_{\mathbb{P}*}(p_X^*F' \otimes p_X^*E_1 \otimes p_{\mathbb{P}}^*\mathcal{O}_{\mathbb{P}}(-1)) \\ &\cong H^0(X, F' \otimes E_1) \otimes \mathcal{O}_{\mathbb{P}}(-1) \end{aligned}$$

and the result follows.  $\square$

**Corollary 3.4.**  *$\mathcal{R}_{\gamma*}^i(f^*\mathcal{W}_\xi(E_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)|_{\gamma^{-1}(Z)}) = 0$  for  $i \neq 1$ . Moreover,*

$$\mathcal{R}_{\gamma*}^1(f^*\mathcal{W}_\xi(E_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)|_{\gamma^{-1}(Z)}) \cong \mathcal{O}_Z^{\oplus n_0}.$$

*Proof.* This follows at once from (14) and Proposition 3.3.  $\square$

Now we are in a position to compute the cohomology groups of  $H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^*E_1 \otimes p_{\mathcal{M}_\xi}^*\mathcal{U}_x^*)$  for  $i = 1, 2$ .



**Proposition 3.5.**  $H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) = 0$  for any  $x \in X$ .

*Proof.* The combination of (6), (7) and (12) yields

$$H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) \cong H^2(\gamma^{-1}(Z), f^* \mathcal{W}_\xi(E_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)|_{\gamma^{-1}(Z)}).$$

Using Corollary 3.4 and Lemma 3.1(i), the Leray spectral sequence for the map  $\gamma$  gives

$$\begin{aligned} H^2(\gamma^{-1}(Z), f^* \mathcal{W}_\xi(E_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)|_{\gamma^{-1}(Z)}) &\cong H^1(Z, \mathcal{O}_Z)^{\oplus n_0} \\ &\cong H^1(\mathcal{M}_\eta, \mathcal{O}_{\mathcal{M}_\eta})^{\oplus n_0}. \end{aligned}$$

It is known that  $H^1(\mathcal{M}_\eta, \mathcal{O}_{\mathcal{M}_\eta}) = 0$  [6]. Therefore,

$$H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) = 0.$$

□

We can now prove

**Proposition 3.6.**  $\mathcal{R}^2 = 0$ .

*Proof.* This is an immediate consequence of Proposition 3.5. □

**Proposition 3.7.** For any point  $x \in X$ ,  $\dim H^1(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) = n_0$ .

*Proof.* As in the proof of Proposition 3.5 we conclude that

$$H^1(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) \cong H^0(\mathcal{M}_\eta, \mathcal{O}_{\mathcal{M}_\eta})^{\oplus n_0}.$$

Now  $\mathcal{M}_\eta$  is just the non-singular part of the moduli space of semistable bundles of rank  $n$  and determinant  $\eta$ , and the latter space is complete and normal. So  $\dim H^0(\mathcal{M}_\eta, \mathcal{O}_{\mathcal{M}_\eta}) = 1$ . □

**Corollary 3.8.**  $\mathcal{R}^1$  is a vector bundle of rank  $n_0$ .

The complete identification of  $\mathcal{R}^1$  will be given in the next section.

**Remark 3.9.** Since the fibres of  $\gamma$  are projective spaces, we have  $\gamma_* \mathcal{O}_{\gamma^{-1}(Z)} \cong \mathcal{O}_Z$  and all the higher direct images of  $\mathcal{O}_{\gamma^{-1}(Z)}$  are 0. Hence

$$H^i(Z, \mathcal{O}_Z) \cong H^i(\gamma^{-1}(Z), \mathcal{O}_{\gamma^{-1}(Z)})$$

for all  $i$ . Similarly

$$H^i(\mathbb{P}(\mathcal{U}_x), \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}) \cong H^i(\mathcal{M}_\xi, \mathcal{O}_{\mathcal{M}_\xi}) = 0$$

for  $i > 0$  since  $\mathcal{M}_\xi$  is a smooth projective rational variety. It follows from the proof of Lemma 3.1 that, if we define  $\delta$  as in (11),

$$\delta \geq i + 2 \geq 3 \Rightarrow H^i(Z, \mathcal{O}_Z) = 0.$$

The proof of Proposition 3.5 now gives

$$(15) \quad \delta \geq i + 2 \geq 4 \Rightarrow H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) = 0.$$

**Proposition 3.10.** Suppose that (1) holds and that  $E_1, E_2$  are semistable bundles of rank  $n_0$  and degree  $d_0$ . If  $\delta \geq i + 3 \geq 5$ , then  $H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*) = 0$ .

*Proof.* It follows from (15) that  $R^i = R^{i+1} = 0$ . The result now follows from Proposition 2.3.  $\square$

**Corollary 3.11.** *Suppose that (1) holds and that  $E$  is a semistable bundle of rank  $n_0$  and degree  $d_0$ . Then*

$$H^2(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(E))) = 0$$

*except possibly when  $g = 3, n = 2, 3, 4$ ;  $g = 4, n = 2$ ;  $g = 5, n = 2$ .*

*Proof.* Take  $E_1 = E_2 = E$  and  $i = 2$  in Proposition 3.10. We need to show that  $\delta \geq 5$ . In fact it follows from (12) that the exceptional cases are precisely those for which  $\delta < 5$ .  $\square$

#### 4. A DIAGONAL ARGUMENT

Let  $\Delta$  be the diagonal divisor in  $X \times X$ . Pull back the exact sequence

$$0 \longrightarrow \mathcal{O}(-\Delta) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

to  $X \times \mathcal{M}_\xi \times X$  and tensor it with  $p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*$ . Now, the direct image sequence for the projection  $p_3$  gives the following exact sequence over  $X$

$$\begin{aligned} \longrightarrow \mathcal{R}^i p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) &\longrightarrow \mathcal{R}^i \longrightarrow \mathcal{R}^i p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) \\ (16) \quad &\longrightarrow \mathcal{R}^{i+1} p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) \longrightarrow \dots \end{aligned}$$

The following proposition will be used in computing  $\mathcal{R}^1$ .

**Proposition 4.1.** *For any  $E_1$ , the direct images of*

$$p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}$$

*have the following description:*

- (1)  $\mathcal{R}^0 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) \cong E_1$
- (2)  $\mathcal{R}^1 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) \cong E_1 \otimes TX$
- (3)  $\mathcal{R}^2 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) = 0$

*where  $TX$  is the tangent bundle of  $X$ .*

*Proof.* Identifying  $\Delta$  with  $X$  we have

$$\mathcal{R}^i p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) \cong \mathcal{R}^i p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes \mathcal{U}_\xi^*).$$

The proposition follows from a result of Narasimhan and Ramanan [16, Theorem 2] that says

$$(17) \quad H^i(\mathcal{M}_\xi, \mathcal{U}_x \otimes \mathcal{U}_x^*) \cong \begin{cases} \mathbb{C} & \text{if } i = 0, 1 \\ 0 & \text{if } i = 2. \end{cases}$$

For  $i = 0$  the isomorphism is given by the obvious inclusion of  $\mathcal{O}_{\mathcal{M}_\xi}$  in  $\mathcal{U}_x \otimes \mathcal{U}_x^*$  and therefore globalises to give  $\mathcal{R}^0 p_{X*}(\mathcal{U}_\xi \otimes \mathcal{U}_\xi^*) \cong \mathcal{O}_X$ . Similarly for  $i = 1$  the isomorphism is given by the infinitesimal deformation map of  $\mathcal{U}_\xi$  regarded as a family of bundles over  $\mathcal{M}_\xi$  parametrised by  $X$ ; this globalises to  $\mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes \mathcal{U}_\xi^*) \cong TX$ .  $\square$

Propositions 4.1 and 2.4 and the exact sequence (16) together give the following exact sequence of direct images

$$\begin{aligned} 0 \longrightarrow E_1 \longrightarrow \mathcal{R}^1 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) \longrightarrow \mathcal{R}^1 \xrightarrow{\alpha} E_1 \otimes TX \\ (18) \qquad \qquad \longrightarrow \mathcal{R}^2 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) \longrightarrow \dots \end{aligned}$$

For any  $x \in X$  we have the cohomology exact sequence

$$\longrightarrow H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*(-x)) \longrightarrow H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)$$

$$(19) \longrightarrow H^i(\mathcal{M}_\xi, \mathcal{U}_x \otimes \mathcal{U}_x^*) \otimes (E_1)_x \longrightarrow H^{i+1}(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*(-x)),$$

where  $(E_1)_x$  is the fibre of  $E_1$  at  $x$ .

By (5),  $p_{X*}(\mathcal{U}_\xi \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) = 0$ . So the Leray spectral sequence for  $p_X$  gives

$$H^1(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) \cong H^0(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*))$$

and

$$H^1(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*(-x)) \cong H^0(X, \mathcal{R}^1 p_{1*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)(-x)).$$

Since  $\mathcal{U}_x$  is simple [16, Theorem 2], (19) gives the exact sequence

$$\begin{aligned} 0 \longrightarrow (E_1)_x \longrightarrow H^0(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)(-x)) \\ \longrightarrow H^0(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)) \longrightarrow \dots \end{aligned}$$

This implies that  $\mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)$  has torsion at  $x$ . Now from (17) we conclude that  $\mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)$  is a torsion sheaf, and hence

$$H^1(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)(-x)) = H^1(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)) = 0.$$

The Leray spectral sequence for  $p_X$  now yields

$$(20) \quad H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) \cong H^0(X, \mathcal{R}^2 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*))$$

and

$$(21) \quad H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*(-x)) \cong H^0(X, \mathcal{R}^2 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)(-x)).$$

Now from (17) it follows that  $\mathcal{R}^2 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*)$  is a torsion sheaf, and from Proposition 3.5 and (20) that its space of sections is 0. So  $\mathcal{R}^2 p_{X*}(\mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*) = 0$  and by (21) we have

$$(22) \quad H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* E_1 \otimes p_{\mathcal{M}_\xi}^* \mathcal{U}_x^*(-x)) = 0.$$

**Proposition 4.2.**  $\mathcal{R}^2 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) = 0$ .

*Proof.* This follows at once from (22). □

**Proposition 4.3.**  $\mathcal{R}^1 \cong E_1 \otimes TX$ .

*Proof.* By Corollary 3.8,  $\mathcal{R}^1$  is a vector bundle of rank  $n_0$ . Moreover Proposition 4.2 implies that the map  $\alpha : \mathcal{R}^1 \longrightarrow E_1 \otimes TX$  in the exact sequence (18) is surjective. Therefore  $\alpha$  must be an isomorphism, and the proof is complete.  $\square$

## 5. AN INVERSION FORMULA

We are now ready to prove our inversion formula.

**Theorem 5.1.** *Suppose that (1) holds and that  $E$  is a semistable bundle of rank  $n_0$  and degree  $d_0$ . Then*

$$E \cong \mathcal{R}^1 p_{X*}(p_{\mathcal{M}_\xi}^* \mathcal{W}_\xi(E) \otimes \mathcal{U}_\xi^* \otimes p_X^* K_X).$$

*Proof.* We consider the Leray exact sequence for the composite

$$p_X \circ p_{23} = p_3 : X \times \mathcal{M}_\xi \times X \longrightarrow X.$$

Note that  $\mathcal{R}^i p_{23*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E \otimes p_{23}^* \mathcal{U}_\xi^*) = 0$  for  $i \geq 1$  by (1), and

$$\begin{aligned} p_{23*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E \otimes p_{23}^* \mathcal{U}_\xi^*) &\cong p_{\mathcal{M}_\xi}^*(p_{\mathcal{M}_\xi*}(\mathcal{U}_\xi \otimes p_X^* E)) \otimes \mathcal{U}_\xi^* \\ &\cong p_{\mathcal{M}_\xi}^* \mathcal{W}_\xi(E) \otimes \mathcal{U}_\xi^*. \end{aligned}$$

Now the Leray spectral sequence gives

$$\begin{aligned} \mathcal{R}^1 p_{X*}(p_{\mathcal{M}_\xi}^* \mathcal{W}_\xi(E) \otimes \mathcal{U}_\xi^*) &\cong \mathcal{R}^1 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* E \otimes p_{23}^* \mathcal{U}_\xi^*) \\ &\cong E \otimes TX \end{aligned}$$

by Proposition 4.3. Tensoring with  $K_X$  now gives the result.  $\square$

## 6. INFINITESIMAL DEFORMATIONS

In this section we turn to the computation of the infinitesimal deformations of the generalised Picard bundle.

**Theorem 6.1.** *Suppose that (1) holds and that  $E_1, E_2$  are semistable bundles of rank  $n_0$  and degree  $d_0$ . Then*

$$H^0(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*) \cong H^0(X, E_1 \otimes E_2^*).$$

*Proof.* By Propositions 2.3 and 2.4, we have

$$H^0(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*) \cong H^0(X, \mathcal{R}^1 \otimes E_2^* \otimes K_X).$$

From Proposition 4.3 it follows immediately that

$$H^0(X, \mathcal{R}^1 \otimes E_2^* \otimes K_X) \cong H^0(X, E_1 \otimes E_2^*)$$

and hence the proof is complete.  $\square$

**Corollary 6.2.** *If  $E$  is semistable and simple of rank  $n_0$  and degree  $d_0$ , then*

$$H^0(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(E))) \cong \mathbb{C}.$$

*In other words, the vector bundle  $\mathcal{W}_\xi(E)$  is simple.*

*Proof.* Take  $E_1 = E_2 = E$  in the theorem.  $\square$

The following theorem now gives the infinitesimal deformations of  $\mathcal{W}_\xi(E)$ .

**Theorem 6.3.** *Suppose that (1) holds. For any semistable bundle  $E$  of rank  $n_0$  and degree  $d_0$ , the space of infinitesimal deformations of the vector bundle  $\mathcal{W}_\xi(E)$ , namely  $H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(E)))$ , is canonically isomorphic to  $H^1(X, \text{End}E)$ . In particular, if  $E$  is also simple,*

$$\dim H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(E))) = n_0^2(g-1) + 1.$$

*Proof.* Let  $E_1 = E_2 = E$ . From Propositions 2.3 and 3.6 we obtain an isomorphism

$$(23) \quad H^1(X, \mathcal{R}^1 \otimes E^* \otimes K_X) \cong H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(E)))$$

From Proposition 4.3 we have

$$\begin{aligned} H^1(X, \mathcal{R}^1 \otimes E^* \otimes K_X) &\cong H^1(X, E \otimes TX \otimes E^* \otimes K_X) \\ &\cong H^1(X, \text{End}E). \end{aligned}$$

Hence

$$H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(E))) \cong H^1(X, \text{End}E)$$

as required. The formula for the dimension follows from Riemann-Roch.  $\square$

**Remark 6.4.** From the proof of Theorem 6.3 we see that

$$H^1(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*) \cong H^1(X, E_1 \otimes E_2^*)$$

for any semistable  $E_1, E_2$  of rank  $n_0$  and degree  $d_0$ . In particular, if  $E_1, E_2$  are stable and not isomorphic, we have  $H^0(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*) = 0$  by Theorem 6.1 and hence

$$\dim H^1(\mathcal{M}_\xi, \mathcal{W}_\xi(E_1) \otimes \mathcal{W}_\xi(E_2)^*) = n_0^2(g-1).$$

## 7. FAMILY OF DEFORMATIONS

In this section we investigate local and global deformations of the generalised Picard bundles constructed above.

First suppose that  $(n_0, d_0) = 1$  and let  $\mathcal{U}(n_0, d_0)$  be a universal bundle over  $X \times \mathcal{M}(n_0, d_0)$ . Now consider  $X \times \mathcal{M}(n_0, d_0) \times \mathcal{M}_\xi$  and define

$$\widehat{\mathcal{U}}_\xi := p_{12}^* \mathcal{U}(n_0, d_0) \otimes p_{13}^* \mathcal{U}_\xi,$$

and

$$\widehat{\mathcal{W}}_\xi := p_{23*}(\widehat{\mathcal{U}}_\xi)$$

By (1) we have  $\mathcal{R}^i p_{23*}(\widehat{\mathcal{U}}_\xi) = 0$  for  $i \neq 0$  and so  $\widehat{\mathcal{W}}_\xi$  is locally free. Moreover

$$\widehat{\mathcal{W}}_\xi|_{\{E\} \times \mathcal{M}_\xi} \cong \mathcal{W}_\xi(E),$$

so  $\widehat{\mathcal{W}}_\xi$  is a family of deformations of  $\mathcal{W}_\xi(E)$ .

**Theorem 7.1.** *The family  $\widehat{\mathcal{W}}_\xi$  is injectively parametrised and is locally complete at every point  $E_0 \in \mathcal{M}(n_0, d_0)$ .*

*Proof.* The injectivity follows from Theorem 5.1. It remains to prove that the infinitesimal deformation map is an isomorphism at every  $E_0$ . By Corollary 6.2,  $\mathcal{W}_\xi(E_0)$  is simple, so possesses a local analytic moduli space  $S$ . It follows that there exists a neighbourhood  $U$  of  $E_0$  in  $\mathcal{M}(n_0, d_0)$  with respect to the analytic topology and a holomorphic map

$$\phi : U \longrightarrow S$$

such that  $\phi(E) \cong \mathcal{W}_\xi(E)$  for all  $E \in U$ . By injectivity the image of  $\phi$  has dimension

$$\dim \mathcal{M}(n_0, d_0) = n_0^2(g-1) + 1$$

at every point. On the other hand, by Theorem 6.3, we know that the Zariski tangent space to  $S$  at  $\mathcal{W}_\xi(E)$  also has dimension  $n_0^2(g-1) + 1$ . It follows that  $S$  is smooth at  $\mathcal{W}_\xi(E)$ . Hence, by Zariski's Main Theorem,  $\phi$  maps  $U$  isomorphically onto an open subset of  $S$ , and in particular the differential  $d\phi$  (which coincides with the infinitesimal deformation map) is an isomorphism at  $E_0$ .  $\square$

**Remark 7.2.** When  $(n_0, d_0) \neq 1$ , we no longer have a universal bundle  $\mathcal{U}(n_0, d_0)$ . However, for any  $E \in \mathcal{M}(n_0, d_0)$ , there exists an étale neighbourhood of  $E_0$  over which a universal bundle does exist. The argument of Theorem 7.1 now goes through to give a family of Picard bundles which is locally complete at  $E_0$ . This family is not injectively parametrised, but it is still true that

$$\mathcal{W}_\xi(E_1) \cong \mathcal{W}_\xi(E_2) \Leftrightarrow E_1 \cong E_2.$$

Theorem 7.1 says that  $\mathcal{M}(n_0, d_0)$  is in some sense a moduli space for the bundles  $\mathcal{W}_\xi(E)$ . Since  $\mathcal{M}(n_0, d_0)$  is irreducible, this implies that all  $\mathcal{W}_\xi(E)$  have the same Hilbert polynomial  $P_{n_0, d_0}$  with respect to the unique polarisation  $\theta_\xi$  of  $\mathcal{M}_\xi$ . We do not know in general that the  $\mathcal{W}_\xi(E)$  possess a good global moduli space. However, if all  $\mathcal{W}_\xi(E)$  are stable with respect to  $\theta_\xi$ , then they belong to the moduli space  $\mathcal{M}_{\mathcal{M}_\xi}(P_{n_0, d_0})$  and indeed to one particular connected component  $\mathcal{M}^0$  of this moduli space. The map  $E \mapsto \mathcal{W}_\xi(E)$  then defines a morphism

$$\phi : \mathcal{M}(n_0, d_0) \longrightarrow \mathcal{M}^0.$$

**Theorem 7.3.** *If  $(n_0, d_0) = 1$  and  $\mathcal{W}_\xi(E)$  is stable with respect to  $\theta_\xi$  for every  $E \in \mathcal{M}(n_0, d_0)$ , then  $\phi$  is an isomorphism.*

*Proof.* By Theorem 7.1,  $\phi$  is an isomorphism onto an open subset of  $\mathcal{M}^0$ . Since  $\mathcal{M}(n_0, d_0)$  is complete, this implies that  $\phi$  is an isomorphism.  $\square$

Theorem 7.3 applies in particular if  $n_0 = 1$ . In this case we can suppose that  $d_0 = 0$ , so that  $\mathcal{M}(n_0, d_0) = J$ , the Jacobian of  $X$ . We know by [4] that  $\mathcal{W}_\xi(\mathcal{O})$  is stable with respect to  $\theta_\xi$ , and the same proof shows that  $\mathcal{W}_\xi(L)$  is stable for any  $L \in J$ .

In this case, we can go a little further. Since we shall want to allow  $X$  and  $\xi$  to vary, we denote the space  $\mathcal{M}^0$  by  $\mathcal{M}_{X, \xi}^0$ . Let  $\Theta$  denote the principal polarisation on  $J$  defined by a theta divisor and let  $\zeta$  denote the polarisation on  $\mathcal{M}_{X, \xi}^0$  defined by the determinant line bundle [3, Section 4].

**Theorem 7.4.** *With respect to the above polarisations, the morphism*

$$\phi : J \longrightarrow \mathcal{M}_{X,\xi}^0$$

*is an isomorphism of polarised varieties.*

*Proof.* We wish to show that the isomorphism  $\phi$  takes  $\zeta$  to a nonzero constant scalar multiple (independent of the curve  $X$ ) of  $\Theta$ .

Take any family of pairs  $(X, \xi)$ , where  $X$  is a connected non-singular projective curve of genus  $g$  and  $\xi$  is a line bundle on  $X$  of degree  $d > n(2g - 2)$ , parametrized by a connected space  $T$ . Consider the corresponding family of moduli spaces  $\mathcal{M}_{X,\xi}^0$  (respectively, Jacobians  $J$ ) over  $T$ , where  $(X, \xi)$  runs over the family. Using the map  $\phi$  an isomorphism between these two families is obtained. The polarisation  $\zeta$  (respectively,  $\Theta$ ) defines a constant section of the second direct image over  $T$  of the constant sheaf  $\mathbb{Z}$  over the family. It is known that for the general curve  $X$  of genus  $g$ , the Neron-Severi group of  $J$  is  $\mathbb{Z}$ . Therefore, for such a curve,  $\phi$  takes  $\zeta$  to a nonzero constant scalar multiple of  $\Theta$ . Since  $T$  is connected, if  $T$  contains a curve with  $NS(J) = \mathbb{Z}$ , then  $\phi$  takes  $\zeta$  to the same nonzero constant scalar multiple of  $\Theta$  for every curve in the family. Since the moduli space of smooth curves of genus  $g$  is connected, the proof is complete.  $\square$

Finally we have our Torelli theorem.

**Corollary 7.5.** *Let  $X$  and  $X'$  be two non-singular algebraic curves of genus  $g \geq 3$  and let  $\xi$  (respectively  $\xi'$ ) be a line bundle of degree  $d > n(2g - 2)$  on  $X$  (respectively  $X'$ ). If  $\mathcal{M}_{X,\xi}^0 \cong \mathcal{M}_{X',\xi'}^0$  as polarised varieties then  $X \cong X'$ .*

*Proof.* This follows at once from Theorem 7.4 and the classical Torelli theorem.  $\square$

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